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Coupling, degeneracy breaking and isolation of Weibel modes in relativistic plasmas: II. Specific examples

R C Tautz[1] **and I Lerche**[2]

[1] Institut für Theoretische Physik, Lehrstuhl IV: Weltraum- und Astrophysik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
[2] Institut für Geowissenschaften, Naturwissenschaftliche Fakultät III, Martin-Luther-Universität Halle, D-06099 Halle, Germany

E-mail: rct@tp4.rub.de and lercheian@yahoo.com

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Abstract
Recently, a general proof was given (Tautz *et al* 2006 *J. Phys. A: Math. Gen.* **39** 13831) that for an asymmetric relativistic particle phase-space distribution function, and in the absence of a homogeneous background magnetic field, any unstable linear Weibel modes are isolated, i.e. restricted to discrete wavenumbers. In this paper, for a specific distribution function consisting of mono-energetic counterstreaming electron and positron beams, growth rates and associated wavenumbers for the isolated modes are calculated, proving the existence of discrete values for unstable wavenumbers. Furthermore, electrostatic and electromagnetic Weibel modes are investigated for monoenergetic counterstreaming plasmas, yielding constraints to the momentum components that have to be fulfilled in order to have unstable wave modes.

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1. Introduction

Because of the intense focus of research of Weibel [1] modes (see also [2–4]) in astrophysical plasmas, we have undertaken a systematic effort to address the corresponding development of such sorts of modes when the plasma distribution functions do not have prescribed symmetries.

The continuing interest in the Weibel instability is underscored in the astrophysical arena by the observations of highly relativistic plasmas, highly energetic particles and their associated radiation (see, e.g., [5] for an introduction to the subject), and by the fact that the classic Weibel instability provides a mode growing exponentially in time but not propagating. The exponential growth finally saturates [6–8] and in the following the magnetic field grows further due to coalescence processes of the current filaments that have been created by the linear growth phase.

Special attention has been given to counterstreaming plasma distributions [9–14] because, as first shown by Fried [15], such distributions are subject to the Weibel instability, with a broad range of applications ranging from the creation of cosmological magnetic fields [16–18] down to local phenomena in the solar system [19–24]. Another topic that has been extensively investigated over the past few years is that of beam–plasma interactions (e.g., Bret *et al* [25–28]).

Astrophysical situations where relativistic effects [29–31] with [32–34] or without magnetic fields play a dominant role are burgeoning as observations uncover more complex and bizarre behaviours that can be accounted for only by inclusion of relativistic attributes, be it in bulk quantities (such as relativistic beaming effects in pulsars, active galactic nuclei and gamma-ray bursts [35–37], see also [38–40] for an overview on the physics of GRB) or in the intrinsic particle distributions themselves (such as cosmic rays).

However, all of the mentioned work involves symmetric or at least gyrotropic (i.e., symmetric around the axis of the counterstream and/or the background magnetic field) distribution functions. In a recent paper (Tautz *et al* [41], which will hereafter be referred to as paper I), linear purely growing instabilities in a relativistic plasma have been investigated in the absence of a homogeneous background magnetic field. Whereas broad wavenumber ranges can be found that permit unstable wave modes for a symmetric particle distribution function, for an asymmetric particle distribution function it has been proven in general that any unstable Weibel modes are isolated. Unless the asymmetry is precisely zero, instead of a broad range of unstable wavenumbers occurring for symmetric distribution functions, there exist only discrete wavenumbers that permit unstable modes. Furthermore, for asymmetric plasmas, electrostatic and electromagnetic wave modes are coupled to each other and, therefore, the degeneracy of the two electromagnetic wave modes is broken.

The question of whether such isolated modes actually exist, however, has not yet been addressed. Only if the equations that determine the growth rate and wavenumber have real and positive solutions, are there unstable modes that are isolated. Here we explore whether isolated Weibel modes exist for even a simple example of a mono-energetic particle distribution function and determine their growth rate as well as their isolated wavenumber values.

This paper is organized as follows: in section 2, unstable electrostatic Weibel modes are explored for counterstreaming mono-energetic plasma beams, showing: (i) their existence; and (ii) analytic expressions for their maximum growth rate and maximum unstable wavenumber, as well as constraints to the momentum components that have to be fulfilled to permit unstable wave modes. In section 3, electromagnetic Weibel modes are investigated for a non-cylindrically symmetric distribution function because, as shown in paper I, the degeneracy of the two electromagnetic wave modes is then broken. In section 4, it is shown that isolated wave modes exist for an asymmetric distribution function, and growth rates as well as the associated wavenumbers are calculated numerically as a function of the Lorentz factor for two interpenetrating electron and positron beams.

The basic dispersion relation [42], described by a 3×3 determinant, was already introduced in the first paper. We therefore do not provide a detailed introduction and refer the interested reader to paper I. Because the investigation is undertaken in the absence of a homogeneous magnetic field in order to evaluate both the coupling effects as well as the degeneracy breaking factors for Weibel-like modes, the propagation direction of the unstable waves will be constrained to the x-direction without any loss of generality.

2. Electrostatic Weibel modes

For symmetric distribution functions, F, with F symmetric in *each* component of the momentum vector $\boldsymbol{p}$, i.e. $F = F\left(p_x^2, p_y^2, p_z^2\right)$ where wave behaviour is $\exp[\mathrm{i}kx + Mkct]$ with M real, the 3×3 dispersion relation decouples and the electrostatic component satisfies

$$k^2 = \sum_a \xi_a^2 \int \mathrm{d}^3 p \frac{\gamma}{p_x - \mathrm{i}M\gamma} \frac{\partial F_a}{\partial p_x} \tag{1a}$$

Because F_a is symmetric in p_x one can write

$$k^2 = 2\sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{p_x\gamma}{p_x^2 + M^2\gamma^2} \frac{\partial F_a}{\partial p_x}, \tag{1b}$$

where $\mathrm{d}^2 p_\perp = \mathrm{d}p_y\,\mathrm{d}p_z$. Furthermore, $\gamma^2 = 1 + p_x^2 + p_y^2 + p_z^2$ denotes the squared Lorentz factor, and $\xi_a^2 = 4\pi n_a e_a^2 \big/ (m_a c^2)$ denotes the inverse squared skin depth, where n_a is the number density of particle species a.

If $\partial F_a/\partial p_x < 0$ everywhere then there are *no* modes to equation (1*a*) with M real and k real because the right-hand side of equation (1*b*) is negative. Consider then that $\partial F_a/\partial p_x$ can change sign in p_x. Integrate equation (1*b*) by parts to obtain

$$k^2 = -2\sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp F_a \frac{\partial}{\partial p_x}\left(\frac{p_x\gamma}{p_x^2 + M^2\gamma^2}\right) \tag{2a}$$

$$= -2\sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp F_a \frac{\gamma_\perp^2}{\gamma} \frac{M^2\gamma^2 - p_x^2}{\left(p_x^2 + M^2\gamma^2\right)^2} \tag{2b}$$

where $\gamma_\perp^2 = 1 + p_y^2 + p_z^2$.

Note that if one takes $M^2 \geqslant 1$ equation (2*b*) has no solutions with k real because the integral is positive. Hence any Weibel modes are limited to $M^2 < 1$, corresponding to the growth rate being subject to the constraint $\Gamma < kc$.

Consider that such a mode exists. Then ask: for what finite values of M can $k = 0$? Equation (2*b*) indicates that these values $M_\star$ are given by

$$M_\star^2 \sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{F_a \gamma_\perp^2 \gamma}{\left(p_x^2 + M_\star^2\gamma^2\right)^2} = \sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{F_a \gamma_\perp^2 \gamma}{\left(p_x^2 + M_\star^2\gamma^2\right)^2} \left(\frac{p_x}{\gamma}\right)^2. \tag{3}$$

The right-hand side integral is always less than the left-hand side integral (and both are positive definite) for all real values of $M_\star$. Hence

$$M_\star^2 = \sum_a \xi_a^2 \int \mathrm{d}^3 p \frac{F_a \gamma_\perp^2 \gamma}{\left(p_x^2 + M_\star^2\gamma^2\right)^2} \left(\frac{p_x}{\gamma}\right)^2 \left(\sum_a \xi_a^2 \int \mathrm{d}^3 p \frac{F_a \gamma_\perp^2 \gamma}{\left(p_x^2 + M_\star^2\gamma^2\right)^2}\right)^{-1}, \tag{4}$$

with $M_\star^2 < 1$ as required.

Consider values of M in the neighbourhood of $M_\star$. Write $M^2 = M_\star^2 + \delta M$. Then

$$k^2 = 2\,\delta M \sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{F_a \gamma_\perp^2 \gamma \left(M_\star^2\gamma^2 - 3p_x^2\right)}{\left(p_x^2 + M_\star^2\gamma^2\right)^3}. \tag{5}$$

Thus, to one side or the other of $M_\star$ the plasma has Weibel modes that grow with time.

If the integral in equation (5) is such that

$$M_\star^2 \sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{F_a \gamma_\perp^2 \gamma^3}{\left(p_x^2 + M_\star^2 \gamma^2\right)^3} \gtrless 3 \sum_a \xi_a^2 \int_0^\infty \mathrm{d}p_x \int \mathrm{d}^2 p_\perp \frac{F_a \gamma_\perp^2 \gamma^3}{\left(p_x^2 + M^2 \gamma^2\right)^3} \left(\frac{p_x}{\gamma}\right)^2 \tag{6}$$

then, at the very least, the neighbourhood $\delta M \gtrless 0$ around $M_\star^2$ (but with $0 < M_\star + \delta M < 1$) provides unstable electrostatic Weibel modes. In equation (6), the symbol '$\gtrless$' corresponds to $\delta M \gtrless 0$, i.e., '$>$' for $\delta M > 0$ and '$<$' for $\delta M < 0$. Therefore, $\delta M \gtrless 0$ also implies $\delta M \neq 0$.

The time dependence of the instability is given by $\exp[kcMt]$, i.e., for unstable modes one requires $kM > 0$. Now, from equation (5) one has $k^2 = 2I(M_\star)\delta M$, where $I(M_\star)$ denotes the integral in equation (5). For $I(M_\star) \gtrless 0$, therefore, $k^2 > 0$ holds when $\delta M \gtrless 0$. Then the growth rate $\Gamma = kcM$ is given by

$$\Gamma = cM\sqrt{2(M^2 - M_\star^2) I(M_\star)}. \tag{7}$$

For a non-relativistic plasma, in which one sets $\gamma = 1$ in equation (1*b*), one has

$$k^2 = -2 \sum_a \xi_a^2 \int \mathrm{d}^3 p\, F_a \frac{\partial}{\partial p_x} \left(\frac{p_x}{p_x^2 + M^2}\right) \tag{8a}$$

$$= -2 \sum_a \xi_a^2 \int \mathrm{d}^3 p\, F_a \frac{M^2 - p_x^2}{\left(p_x^2 + M^2\right)^2}. \tag{8b}$$

Equation (8*a*) does *not* provide an automatic constraint on M in order that Weibel-like modes exist, which is completely the opposite to the full relativistic treatment where $M < 1$ is necessary.

To show that such Weibel-like modes exist is most easily accomplished by considering a double-beam electron–positron plasma with charge neutrality in *each* beam so that one would write

$$F_a = \frac{1}{2\pi}[\delta(p_x - \varpi) + \delta(p_x + \varpi)]\delta\left(p_\perp^2 - \Pi^2\right) \tag{9}$$

thereby preserving the symmetry under $p_i \to -p_i$ for $i \in \{x, y, z\}$. Note that $\int F_a\, \mathrm{d}^3 p = 1$.

Then from equation (2*b*) one has

$$k^2 = -2\xi_\mathrm{e}^2 \frac{M^2 - v_x^2}{\left(M^2 + v_x^2\right)^2} \frac{\gamma_\perp^2}{\gamma^3}, \tag{10}$$

with $\gamma_\perp^2 = 1 + \Pi^2$, $\gamma^2 = \gamma_\perp^2 + \varpi^2$ and $v_x^2 = \varpi^2/\gamma^2$.

The instability rate, Γ, is then given through

$$\Gamma^2 = 2c^2 \xi_\mathrm{e}^2 \frac{\gamma_\perp^2}{\gamma^3} \frac{(1 - w^2) w^2}{(1 + w^2)^2}, \tag{11}$$

where $w = M/v_x$. The instability rate has a maximum on $w^2 = 1/3$ when

$$\Gamma_\mathrm{max}^2 = c^2 \xi_\mathrm{e}^2 \frac{\gamma_\perp^2}{8\gamma^3} \tag{12a}$$

and the associated wavenumber is

$$k_\mathrm{max}^2 = c^2 \xi_\mathrm{e}^2 \frac{3}{4 v_x^2} \frac{\gamma_\perp^2}{\gamma^3} \tag{12b}$$

and $M_\mathrm{max} = v_x/\sqrt{3}$.

All wavenumbers in $0 \leqslant k \leqslant k_\star$ are unstable with $k_\star^2 = 8k_{\max}^2/3$. Note that because $v_x \leqslant 1$ then $M^2 < 1$ as required for an instability.

Non-relativistically one would have

$$\Gamma^2 = 2\xi_e^2 \frac{(1-u^2)u^2}{(1+u^2)^2} \tag{13}$$

with $u = M/|p_x|$, so that $\Gamma_{\max}^2 = \xi_e^2/4$, $M_{\max} = \Gamma_{\max}/\sqrt{3}$ and $k_{\max} = 3\xi_e^2/\left(4\Gamma_{\max}^2\right)$, showing the basic differences between the relativistic and non-relativistic treatments. The range of unstable wavenumbers in the non-relativistic treatment is $0 \leqslant k \leqslant \tilde{k}$, with $\tilde{k}^2 = 2k_\star^2/\varpi^2$.

3. Electromagnetic Weibel modes

The two electromagnetic modes are given by

$$k^2(1+M^2) = -\sum_a \xi_a^2 (g_{x,a} + g_{z,a} + I_{yy,a}) \tag{14a}$$

$$k^2(1+M^2) = -\sum_a \xi_a^2 (g_{x,a} + g_{y,a} + I_{zz,a}). \tag{14b}$$

In general, for an asymmetric plasma, one has

$$I_{(yy,zz),a} = \int \mathrm{d}^3 p \frac{F_a p_{y,z}^2}{\gamma^3\left(p_x^2 + M^2\gamma^2\right)^2}\left[p_x^2\left(\gamma_\perp^2 + 2p_x^2 + 2M^2\gamma^2\right) - M^2\gamma^2\gamma_\perp^2\right] \tag{15a}$$

$$g_{x,a} = \int \mathrm{d}^3 p \frac{1+p_x^2}{\gamma^3} F_a(\boldsymbol{p}) \tag{15b}$$

$$g_{(y,z),a} = \int \mathrm{d}^3 p \frac{p_{y,z}^2}{\gamma^3} F_a(\boldsymbol{p}). \tag{15c}$$

Because $g_{x,a}$ and $g_{y,a}$ are both positive, in order to have an electromagnetic mode with $k^2 > 0$ for M real one requires the necessary constraint $\sum_a I_{yy,a} < 0$ or $\sum_a I_{zz,a} < 0$. Note that such a constraint is *not* sufficient because $g_{x,a}$ and $g_{y,a}$ are real and positive. One requires the necessary and sufficient constraint

$$\sum_a \xi_a^2 I_{yy,a} < -\sum_a \xi_a^2 (g_{x,a} + g_{z,a}) \tag{16a}$$

or

$$\sum_a \xi_a^2 I_{zz,a} < -\sum_a \xi_a^2 (g_{x,a} + g_{z,a}). \tag{16b}$$

Note that, unlike the electrostatic mode, there is no obvious, universal limitation on the value of M in order to obtain $k^2 > 0$. Instead any limitations on M arise from the type of plasma distribution function chosen and are, therefore, plasma specific rather than being universal.

One can, however, note that as $M \to \infty$, both $I_{yy,a}$ and $I_{zz,a}$ vary as M^{-2} so that equations (16*a*) and (16*b*) *cannot* be satisfied. There is thus a maximum limit on M in order to have $k^2 > 0$, which limit varies as the plasma distribution functions are varied.

To exhibit the degeneracy breaking again consider a double-beam neutral electron–positron plasma with

$$F_a = \frac{\Pi_y \Pi_z}{2}[\delta(p_x - \varpi) + \delta(p_x + \varpi)]\delta\left(p_y^2 - \Pi_y^2\right)\delta\left(p_z - \Pi_z^2\right) \tag{17}$$

so that F_a is symmetric under $p_i \to -p_i$ for $i \in \{x, y, z\}$ and $\int F_a \, \mathrm{d}^3 p = 1$.

In this situation, the condition (inequalities (16*a*) and (16*b*)) that one obtains $k^2 > 0$ for real M can be written as

$$\frac{(\Pi_y^2; \Pi_z^2)}{\gamma^3(\varpi^2 + M^2\gamma^2)^2}\left[\varpi^2\left(\gamma_\perp^2 + 2\varpi^2 + 2M^2\gamma^2\right) - M^2\gamma^2\gamma_\perp^2\right] \leqslant -\frac{1 + \varpi^2 + (\Pi_y^2; \Pi_z^2)}{\gamma^3} \tag{18}$$

with $\gamma^2 = 1 + \varpi^2 + \Pi_y^2 + \Pi_z^2$, and $\gamma_\perp^2 = 1 + \Pi_y^2 + \Pi_z^2 = \gamma^2 - \varpi^2$.

Thus one of the following conditions must be fulfilled: either

$$\Pi_y^2\left[\varpi^2\left(\gamma_\perp^2 + 2\varpi^2 + 2M^2\gamma^2\right) - M^2\gamma^2\gamma_\perp^2\right] < -\left(1 + \varpi^2 + \Pi_z^2\right)(\varpi^2 + M^2\gamma^2)^2 \tag{19a}$$

or

$$\Pi_z^2\left[\varpi^2\left(\gamma_\perp^2 + 2\varpi^2 + 2M^2\gamma^2\right) - M^2\gamma^2\gamma_\perp^2\right] < -\left(1 + \varpi^2 + \Pi_y^2\right)(\varpi^2 + M^2\gamma^2)^2. \tag{19b}$$

Equations (19*a*) and (19*b*) are quadratic inequalities in M^2, requiring that M^2 either exceeds a minimum of M_{min}^2, or be less than a maximum of M_{max}^2, respectively. For instance, inequality (19*b*) can be written as the condition

$$M^4\gamma^2\left(\gamma^2 - \Pi_y^2\right) + M^2\left(2\varpi^2\gamma^2 - \Pi_y^2\gamma_\perp^2\right) + \varpi^2\left(\varpi^2 + \Pi_y^2\right) < 0 \tag{20}$$

showing that it is necessary to have

$$\Pi_y^2\gamma_\perp^2 > 2\varpi^2\gamma^2 \tag{21}$$

in order to have any solution with M^2 real. And when inequality (21) is satisfied then a minimum M_{min}, and a maximum M_{max} occur at

$$\begin{aligned} M_{\text{min,max}}^2\left[2\gamma^2\left(\gamma^2 - \Pi_y^2\right)\right] &= \gamma^2\left(\Pi_y^2\gamma_\perp^2 - \varpi^2\gamma^2\right) \\ &\quad \mp \left[\left(\Pi_y^2\gamma_\perp^2 - \varpi^2\gamma^2\right)^2 - 4\gamma^6\varpi^2\left(\varpi^2 + \Pi_y^2\right)\left(\gamma^2 - \Pi_y^2\right)\right]^{1/2} \end{aligned} \tag{22}$$

which requires the further condition be satisfied that

$$\left(\Pi_y^2\gamma_\perp^2 - 2\varpi^2\gamma^2\right)^2 \geqslant 4\gamma^6\varpi^2\left(\varpi^2 + \Pi_y^2\right)\left(\gamma^2 - \Pi_y^2\right). \tag{23}$$

When both equations (21) and (23) are obeyed then

$$\begin{aligned} k^2 = \frac{2\xi_e^2}{\gamma^3\left(p_x^2 + M^2\gamma^2\right)^2}(1 + M^2)^{-1}\Big\{&\Pi_y^2\left[\varpi^2\left(\gamma_\perp^2 + 2\varpi^2 + 2M^2\gamma^2\right) - M^2\gamma^2\gamma_\perp^2\right] \\ &+ \left(1 + \varpi^2 + \Pi_z^2\right)\left(p_x^2 + M^2\gamma^2\right)^2\Big\}. \end{aligned} \tag{24}$$

The growth rate, $\Gamma = kMc$, of the wave then satisfies

$$\Gamma^2 = \frac{M^2}{1 + M^2}\frac{2c^2\xi_e^2}{\left(p_x^2 + M^2\gamma^2\right)^2}\left(M_{\text{max}}^2 - M^2\right)\left(M^2 - M_{\text{min}}^2\right) \tag{25}$$

showing a peak value of M lying in $]M_{\text{min}}, M_{\text{max}}[$. If either inequality (21) or inequality (23) are *not* satisfied then there are no purely growing electromagnetic modes for the two-stream electron–positron symmetric plasma.

For other than a symmetric, cold, beam plasma one must evaluate the integrals in $I_{yy,a}$ and $I_{zz,a}$ to determine where, or whether, a range of positive M values can exist allowing $k^2 > 0$.

For inequality (19*b*), the same sense of argument is obtained as that just presented for inequality (19*b*). There is no need to repeat the argument.

4. Isolated modes

In this section, an asymmetric distribution function is adopted, for which neither $F(\boldsymbol{p}) = F(-\boldsymbol{p})$ nor $F(p_i, p_j, p_k) = F(-p_i, p_j, p_k)$ with $i, j, k \in \{x, y, z\}$ is fulfilled. Therefore, all nine elements of the 3×3 determinant describing the dispersion relation are non-vanishing. As proven in paper I, for an asymmetric distribution function all unstable Weibel modes are restricted to discrete wavenumbers. In order to calculate actual growth rates, however, an analytic investigation is complicated and, therefore, a numerical approach is favoured.

The asymmetric distribution function is constructed of two interpenetrating particle beams, one of which is formed only by electrons, whereas the other one carries only positrons. Therefore the following form for the electron and positron distribution functions are adopted, respectively:

$$F_{\mathrm{e}} = \delta(p_x - \varpi_x)\delta(p_y - \varpi_y)\delta(p_z - \varpi_z) \tag{26a}$$

$$F_{\mathrm{p}} = \delta(p_x + \varpi_x)\delta(p_y + \varpi_y)\delta(p_z + \varpi_z). \tag{26b}$$

Due to the requirement of charge neutrality, the normalized momenta ϖ_j, with $j \in \{x, y, z\}$, of electrons and positrons have to be equal. The symmetric and anti-symmetric parts of the distribution functions then yield

$$F_{\mathrm{e}}^{S,A} = \pm F_{\mathrm{p}}^{S,A} = \frac{1}{2}\left[\prod_j \delta(p_j - \varpi_j) \pm \prod_j \delta(p_j + \varpi_j)\right], \tag{27}$$

where the '+' and '−' signs refer to F^S and F^A, respectively.

4.1. The elements of the dispersion relation

The elements occurring in the 3×3 determinant of equation (12) of paper I,

$$\begin{vmatrix} k^2 - \Lambda & C_y & C_z \\ -C_y & k^2 + Y & D \\ -C_z & D & k^2 + Z \end{vmatrix} = 0 \tag{28}$$

can then easily be evaluated and are listed in appendix A. As shown in paper I, the real and imaginary parts of the determinant from equation (28) are polynomials in k^2 of order 3 and 2, respectively and can, therefore, be cast in the form

$$0 = a(M)\kappa^6 + b(M)\kappa^4 + d(M)\kappa^2 + f(M) \tag{29a}$$

$$0 = \tilde{b}(M)\kappa^4 + \tilde{d}(M)\kappa^2 + \tilde{f}(M), \tag{29b}$$

where, in contrast to paper I, the factors $a, b, \tilde{b}, d, \tilde{d}, f$, and $\tilde{f}$ consist of the normalized factors from appendix A and are listed in appendix B. For later use, the wave vector has been written as $k^2 = 2\kappa^2\xi_{\mathrm{e}}^2$, with $\xi_{\mathrm{e}}^2 = \omega_{\mathrm{p,e}}^2/c^2$, where $\omega_{\mathrm{p,e}}$ denotes the electron plasma frequency.

Introducing

$$\eta(M) = \tilde{f}(M)b(M)\tilde{b}(M) - \tilde{d}(M)\tilde{f}(M)a(M) - f(M)\tilde{b}(M)^2 \tag{30a}$$

$$\zeta(M) = \tilde{d}(M)^2 a(M) - \tilde{f}(M)a(M)\tilde{b}(M) - \tilde{d}(M)b(M)\tilde{b}(M) + d(M)\tilde{b}(M)^2 \tag{30b}$$

which correspond to the factors $\eta(M)$ and $\zeta(M)$ from equations (17a) and (17b) of paper I multiplied by $\tilde{b}(M)^2$, then allows one to write equation (18) of paper I in the form

$$\tilde{b}(M)\eta(M)^2 + \tilde{d}(M)\eta(M)\zeta(M) + \tilde{f}(M)\zeta(M)^2 = 0, \tag{31}$$

thereby avoiding the zeros of $\tilde{b}(M)$ when numerically searching for solutions of equation (31).

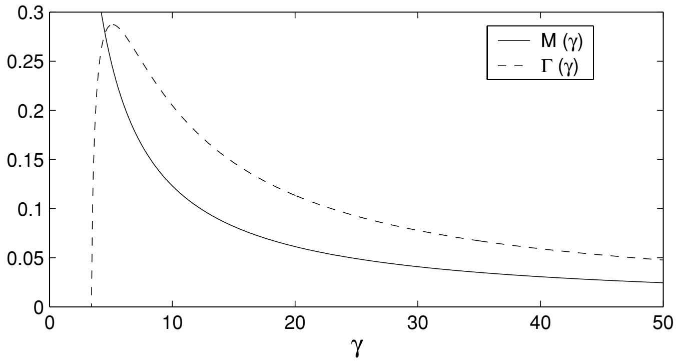

Figure 1. The isolated values for M and the associated growth rate $\Gamma = \sqrt{2}\kappa M$ (normalized to the electron plasma frequency) as a function of the Lorentz factor γ. For each value of γ, a single value for the wavenumber and the growth rate can be obtained, corresponding to isolated, purely growing ($\Gamma > 0$) Weibel modes.

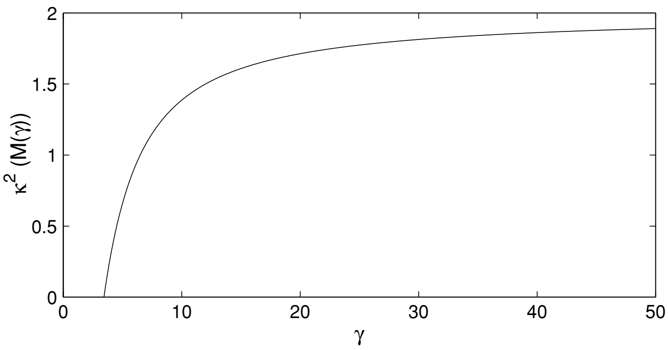

Figure 2. The squared normalized wavenumber κ^2 for the isolated Weibel modes as a function of the Lorentz factor γ.

4.2. Numerical evaluation

For simplification, $\varpi_x = \varpi_y = \varpi_z \equiv \varpi$ is assumed. In this case there is a single free parameter, namely the Lorentz factor γ, that has to be equal for electrons and positrons in order to assure the charge neutrality. The momentum is then obtained as $\varpi = [(\gamma^2 - 1)/3]^{1/2}$.

The polynomial equation (31) can be solved numerically as a function of the Lorentz factor, yielding discrete values for the growth rate Γ and the associated (normalized) wavenumber κ that describe isolated wave modes. The wavenumber is evaluated from the condition $\kappa^2 = \eta'(M)/\zeta'(M)$, where the factors $\eta(M) = \eta'(M)\xi_{\mathrm{e}}^6$ and $\zeta(M) = \zeta'(M)\xi_{\mathrm{e}}^4$ are given from equations (30*a*) and (30*b*).

One of the various solutions for M as a function of the Lorentz factor γ is shown in figure 1, together with the (normalized) growth rate $\Gamma = \sqrt{2}\kappa M$. The growth rate has a maximum for counterstreams with a Lorentz factor of $\gamma \simeq 5.11$. In figure 2, the associated (squared) wavenumber values are shown, which are positive provided $\gamma \gtrsim 3.43$. Therefore, for values of γ greater than a specific value (dependent on the details of the distribution function), this illustration proves that isolated, purely growing Weibel modes exist for the distribution function of equations (26*a*) and (26*b*).

There are other solutions of equation (31) apart from that shown in figure 1. However, these solutions correspond to negative κ^2 and are, therefore, of no value to the investigation of unstable isolated Weibel modes.

5. Summary and discussion

With this series of two papers, unstable relativistic Weibel modes have been investigated in the absence of a homogeneous background magnetic field.

In the first paper (Tautz *et al* [41]), different types of momentum symmetry in the distribution function have been investigated, i.e., symmetry in each momentum component and total momentum symmetry. Next, for a non-cyndrically symmetric distribution function, it was shown that the degeneracy of the two electromagnetic wave modes was broken. Furthermore, for a totally asymmetric distribution function, a general proof was given that any unstable Weibel modes are isolated, i.e., restricted to discrete values for the growth rate and the associated wavenumber.

In this second paper, three subjects have been addressed: first, unstable electrostatic Weibel modes have been investigated and it has been shown that, for relativistic plasmas, growth rate values are restricted to $\Gamma < ck$ (i.e., $M < 1$).

Second, for a distribution function that is asymmetric in the plane perpendicular to the direction of wave propagation, it was shown that the degeneracy of the two electromagnetic wave modes is broken. Furthermore, analytic constraints to the momentum components were derived that have to be fulfilled in order to obtain unstable wave modes.

Third, according to the general proof given in paper I that, for an asymmetric distribution function, any unstable Weibel modes are isolated, a simple asymmetric distribution function with mono-energetic, interpenetrating electron and positron beams was adopted for illustration purposes. The existence of isolated growth rate values and their associated wavenumbers as a function of the beams' Lorentz factor as well as values for the maximum growth rate were calculated numerically. Furthermore, it was shown that these isolated unstable Weibel modes occur for relativistic beams with $\gamma > 1$ (if the momentum components of the beams are equal in all three directions, this constraint yields $\gamma \gtrsim 3.43$). The determination of all the classes of asymmetric distribution functions that permit isolated Weibel modes is beyond the scope of the present paper. The simple illustrations used here show, however, that such isolated modes exist.

We conclude, therefore, that unstable isolated wave modes can be of importance *especially* in relativistic beams that are, for example, present in astrophysical jet sources such as gamma-ray bursts (GRB) and active galactic nuclei (AGN).

It has not escaped our attention that the transition from isolated Weibel modes to the standard continuum representation as one treats with a plasma that is either asymmetric or symmetric, respectively, is a problem requiring further elucidation than we give here. As already mentioned in paper I, it is possible that the countable set of discrete modes is comparable to that obtained from multi-beam plasmas. In such cases the count of discrete modes in each momentum interval tends to infinity as the number of beams becomes unlimitedly large. This leads, therefore, to the continuum representation of the modes. It is not clear by now, whether the same is true for the isolated Weibel-type modes when an initially asymmetric distribution function becomes symmetric in character. If such was the case, however, the limit of an increasing number of discrete values is still unlikely to yield a whole interval, as the referee has noted: an increasing number of discrete values can only yield an infinite *countable* set that is in bijection with $\mathbb{N}$ (the set of integer numbers), and not an uncountable set like $\mathbb{R}$ (the set of real numbers). The problem is that a real interval is *not* in bijection with $\mathbb{N}$ (it is 'larger'). It seems, therefore, that the transition cannot be smooth. We have this problem under active investigation at the moment and results will be reported in due course.

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Appendix A. The elements of the determinant from equation (28)

The elements needed in equation (28) are defined in equations (13) of paper I and yield for the distribution functions from equations (26*a*) and (26*b*):

$$\Lambda^R = \frac{\gamma_\perp^2}{\gamma\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\varpi_x^2 - M^2\gamma^2\right) \tag{A.1a}$$

$$\Lambda^I = \frac{2M\varpi_x}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\gamma^2 - \varpi_x^2\right) \tag{A.1b}$$

$$C_{y,z}^R = \frac{\varpi_{y,z}}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\varpi_x^2 - M^2\gamma_\perp^2\right) \tag{A.1c}$$

$$C_{y,z}^I = \frac{2M\varpi_x\varpi_{y,z}}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}(1 + M^2) \tag{A.1d}$$

$$D^R = -\frac{\varpi_y\varpi_z}{\gamma^3} + \frac{\varpi_y\varpi_z}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\varpi_x^2 - M^2\gamma_\perp^2\right) \tag{A.1e}$$

$$D^I = \frac{2M\varpi_x\varpi_y\varpi_z}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}(1 + M^2) \tag{A.1f}$$

$$Y^R = \frac{1 + \varpi_x^2 + \varpi_z^2}{\gamma^3} + \frac{\varpi_y^2\gamma_\perp^2}{\gamma^3\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\varpi_x^2 - M^2\gamma^2\right) \tag{A.1g}$$

$$Y^I = \frac{2M\varpi_x\varpi_y^2}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}(1 + M^2) \tag{A.1h}$$

$$Z^R = \frac{1 + \varpi_x^2 + \varpi_y^2}{\gamma^3} + \frac{\varpi_z^2\gamma_\perp^2}{\gamma^3\left(\varpi_x^2 + M^2\gamma^2\right)^2}\left(\varpi_x^2 - M^2\gamma^2\right) \tag{A.1i}$$

$$Z^I = \frac{2M\varpi_x\varpi_z^2}{\left(\varpi_x^2 + M^2\gamma^2\right)^2}(1 + M^2), \tag{A.1j}$$

where $\gamma = \left(1 + \varpi_x^2 + \varpi_y^2 + \varpi_z^2\right)^{1/2}$ and $\gamma_\perp = \left(1 + \varpi_y^2 + \varpi_z^2\right)^{1/2}$. Furthermore, all elements are normalized to $\xi_e^2 = \omega_{p,e}^2/c^2$, where $c/\omega_{p,e}$ denotes the electron skin depth.

Appendix B. The elements of the polynomials determining *M*

The elements entering the polynomials in κ from equations (29*a*) and (29*b*) are

$$a(M) = 1 \tag{B.1a}$$

$$b(M) = -\left(\Lambda^R + Y^R + Z^R\right) \tag{B.1b}$$

$$d(M) = \left(C_y^R\right)^2 - \left(C_y^I\right)^2 - \left(C_z^I\right)^2 + \left(C_z^R\right)^2 - (D^R)^2 + (D^I)^2 \\ - \Lambda^R Y^R + \Lambda^I Y^I + (\Lambda^I - Y^I)Z^I - (\Lambda^R - Y^R)Z^R \tag{B.1c}$$

$$f(M) = 2\left(C_y^R C_z^I + C_z^R C_y^I\right)D^I - 2\left(C_y^R C_z^R - C_y^I C_z^I\right)D^R + \left[(D^R)^2 - (D^I)^2 - 2D^I D^R\right]\Lambda^R \\ + \left[\left(C_z^R\right)^2 - \left(C_z^I\right)^2\right]Y^I + 2C_z^I C_z^R Y^R \\ + \left[\left(C_y^R\right)^2 - \left(C_y^I\right)^2 - \Lambda^R Y^R + \Lambda^I Y^I\right]Z^I \\ + \left(2C_y^I C_y^R - \Lambda^R Y^I - \Lambda^I Y^R\right)Z^R \tag{B.1d}$$

$$\tilde{b}(M) = -(\Lambda^I + Y^I + Z^I) \tag{B.1e}$$

$$\tilde{d}(M) = 2\left(C_y^I C_y^R + C_z^I C_z^R - D^I D^R\right) - \Lambda^R(Y^I + Z^I) - \Lambda^I(Y^R + Z^R) + Y^R Z^I + Y^I Z^R \tag{B.1f}$$

$$\tilde{f}(M) = 2\left(C_y^I C_z^I - C_y^R C_z^R\right)D^I - 2\left(C_y^R C_z^I + C_y^I C_z^R\right)D^R + [(D^R)^2 - (D^I)^2]\Lambda^I \\ + 2D^R D^I \Lambda^R + \left[\left(C_z^R\right)^2 - \left(C_z^I\right)^2\right]Y^I + 2C_z^R C_z^I Y^R + \left[\left(C_y^R\right)^2 - \left(C_z^I\right)^2\right]Z^I \\ + 2C_y^R C_y^I Z^R - (\Lambda^R Y^R - \Lambda^I Y^I)Z^I - (\Lambda^R Y^I + \Lambda^I Y^R)Z^R \tag{B.1g}$$

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